Quasilinear thermalization of collision-poor plasmas by noncollective fluctuations

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The observed Maxwellian velocity distribution functions in plasmas and the fact that the rate of elastic electron-electron is many orders of magnitude smaller than the electron plasma frequency has been a long-standing puzzle. Here, we present a mechanism for the efficient thermalization in collision-poor unmagnetized plasmas that resolves this puzzle. The competition between the momentum losses of plasma particles by spontaneously emitting high-frequency non-collective fluctuations and the momentum diffusion of these particles in their self-generated fluctuating electric field fluctuations provides the Maxwellian particle distribution function. The mechanism is self-regulating, providing electron temperatures of about 10^7 K, and is applicable to fully-ionized plasmas with electron densities below 10^{27} cm^{-3}.

In many space plasmas including solar-type stars the plasma parameter $g = 3\nu_{ee}/\omega_{p,e}$, defined by the electron-electron elastic collision rate ν_{ee} and the electron plasma frequency $\omega_{p,e}$ is many orders of magnitude smaller than unity for reported electron temperatures T_e and number densities n_e (Fig. 1 in [1]). These systems are classified as collision-poor or kinetic Vlasov plasmas in which the interactions with electromagnetic fields characterized by $\omega_{p,e} \simeq 5.64 \times 10^4 n_e^{1/2}$ strongly dominate the collision rate $\nu_{ee} \simeq 13.72\, n_e T_e^{-3/2}$ [2]. Elastic Coulomb as well as Møller and Bhabha scattering therefore cannot maintain thermal Maxwell particle distribution functions in these plasmas in contrast to collision-dominated plasmas [3, 4]. Efficient Maxwellization of these plasmas allow the often used theoretical description of these systems with magnetohydrodynamical (MHD) equations derived from taking velocity moments of the underlying particle kinetic equations. Here the temperature enters via the ideal or adiabatic equations of state.

Observationally, however, there is clear evidence for the presence of thermal particle distribution functions in cosmic collision-poor plasmas. This is strongly supported by in-situ plasma measurements of the solar wind. The solar wind plasma is the only cosmic plasma where detailed in-situ satellite observations of plasma properties are available [5–7]. The plasma parameters of the solar wind are similar to the fully ionized phases of the interstellar gas, so that the same plasma physical processes leading to thermalization should also operate in these systems. Although the detailed plasma relaxation processes are not understood [8], the measured electron and proton distribution functions have been modeled with bi-Maxwellian velocity distributions with different temperatures along and perpendicular to the ordered magnetic field direction $\mathbf{B}_0$, which are special cases of velocity-anisotropic particle distribution functions. The WIND/-SWE satellite [6, 7] has measured finite magnetic field fluctuations only within the colored rhomb-shaped configuration in the parameter plane defined by the temperature anisotropy $A = T_\perp/T_\parallel$ and the parallel plasma beta $\beta_\parallel = 8\pi n_e k_{\rm B} T_\parallel/B_0^2$. Magnetic field fluctuations only occur close to the isotropy and equipartition values $A = 1$ and $\beta_\parallel = 1$, respectively. The solar corona, the partially ionized and fully ionized phases of the interstellar medium in our and other galaxies, as well as active galactic nuclei, the intergalactic medium, the intracluster gas and cosmic voids also are collision-poor plasmas.

It is the purpose of this work to propose an efficient kinetic Maxwellization process in collision-poor unmagnetized plasmas relying on the interaction with the noncollective fluctuations having no dispersion relation. The collective modes, where the complex frequency $\omega(\mathbf{k}) = \omega_R(\mathbf{k}) + \imath\Gamma(\mathbf{k})$ is related to the real-valued wave vector $\mathbf{k}$, in unmagnetized plasmas with isotropic particle distribution have been well studied before including their backreaction on the charged plasma particles. Besides the undamped ($\Gamma = 0$) superluminal electromagnetic waves that cannot resonantly interact with the particles, three damped (with negative $\Gamma < 0$) modes exist: the longitudinal electrostatic waves [9], in electron-ion-plasmas the longitudinal ion sound waves, and transverse aperiodic (with $\omega_R = 0$) fluctuations [10, 11], which oscillate in space but do not propagate. In particular no growing eigenmodes are possible in these plasmas [10, 12–16].

Alternatively, the noncollective fluctuations have unrestricted complex frequencies, and in the following we investigate their generation and back reaction on the particles. It is demonstrated that the quasilinear interactions of charged plasma particles with the spontaneously emitted noncollective fluctuations provide an alternative mechanism to produce thermal particle distribution functions in collision-poor space plasmas. Every plasma spontaneously emits electromagnetic fluctuations due to the random motions of its charged particles causing subsequent particle-fluctuation interactions. These are described by quasilinear transport theory if the ratio of particle number density fluctuations is small compared to the averaged particle density. We follow the derivation of the general quasilinear Fokker-Planck kinetic equation for the gyrophase-averaged plasma particle distri-

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bution functions in magnetized plasmas ([17] – hereafter referred to as paper I) making no restrictions on the energy of the particles and on the frequency of the electromagnetic fluctuations. The derivation is based on Maxwell equations for the electromagnetic fields and on the Klimontovich equation [18] for the charged particles accounting for discrete particle effects and spontaneous emission of fluctuations. The monograph [19] provides an excellent introduction into the combined kinetic theory although this approach adopts nonrelativistic particle energies throughout.

Our investigation here will be fully relativistic. As shown in paper I the inclusion of discrete particle effects breaks the dichotomy of nonlinear kinetic plasma theory divided into the test particle and the test fluctuation approximation because it provides expression of electromagnetic fluctuation spectra in terms of the plasma particle distribution functions. The resulting quasilinear transport equation can be regarded as a determining nonlinear equation for the time evolution of the particle distribution function. The general case of magnetized plasmas has been investigated before in paper I, but has lead to very involved final expressions. In its most general form this nonlinear equation is rather involved and complicated. Therefore it is appropriate to study simpler plasma configurations such as unmagnetized plasmas which is the purpose of the present manuscript. According to Eqs. (78)–(81) of paper I, the general gyrophase-averaged quasilinear transport equation for the spatially uniform phase space distribution function $n_a f_a(\mathbf{p},t)$ of sort a with charge q_a and mass m_a in spherical momentum coordinates $\mathbf{p}=p(\sqrt{1-\mu^2}\cos\phi, \sqrt{1-\mu^2}\sin\phi, \mu)$ reads

$$\frac{\partial f_a}{\partial t}+\frac{1}{2\pi}\int_0^{2\pi}d\phi\, Q_a(\mathbf{r},\mathbf{p},t)=\mathcal{D}_1(f_a)+\mathcal{D}_2(f_a), \quad (1)$$

with the gyrophase-averaged drag and momentum diffusion terms

$$\begin{pmatrix}\mathcal{D}_1(f_a)\\ \mathcal{D}_2(f_a)\end{pmatrix}=-\frac{1}{2\pi n_a}\int d^3k\int d\omega\ \times \int_0^{2\pi}d\phi\,\frac{\partial}{\partial\mathbf{p}}\cdot\begin{pmatrix}\Re\langle\mathbf{K}^*_{\mathbf{k},\omega}(\mathbf{p})\delta N^{a0}_{\mathbf{k},\omega}(\mathbf{p})\rangle\\ \Re\langle\mathbf{K}^*_{\mathbf{k},\omega}(\mathbf{p})A_{\mathbf{k},\omega}(\mathbf{p})\rangle\end{pmatrix}, \quad (2)$$

respectively, involving the Fourier-Laplace transformed stochastic force $\mathbf{K}_{\mathbf{k},\omega}(\mathbf{p})$ and free-particle Klimontovich particle density $\delta N^{a0}_{\mathbf{k},\omega}(\mathbf{p})$ of sort a, while Q_a accounts for sources and sinks of particles.

Here we are primarily interested in the isotropic equilibrium distribution functions $f_a(p)$ determined by the balance of momentum loss of plasma particles by spontaneous emission (described by the drag term) and the momentum diffusion of plasma particles caused by the electromagnetic fluctuations, respectively. Moreover, in stationary plasmas one is interested in temporal averages, where the averaging time $\mathcal{T}$ must be greater than the correlation time of fluctuations for this average to be independent of $\mathcal{T}$ [20, 21]. This leads to $\overline{D}_1(f_a)+\overline{D}_2(f_a)=0$ with

$$\overline{D}_{1,2}(f_a)=\lim_{\mathcal{T}\to\infty}\frac{1}{\mathcal{T}}\int_{-\mathcal{T}/2}^{\mathcal{T}/2}\frac{dt}{2}\int_{-1}^{1}d\mu\,\mathcal{D}_{1,2}(f_a). \quad (3)$$

For isotropic and gyrotropic distribution functions $df_a(p)/d\mathbf{p}=f'_a(p)\,\mathbf{p}/p$. In the case of unmagnetized plasma considered here, we orient the wave vector along the cartesian z-direction $\mathbf{k}=k\mathbf{e}_z$. Likewise, one obtains $A_{\mathbf{k},\omega}(\mathbf{p})=G(p,\mu)[\mathbf{p}\cdot\mathbf{K}_{\mathbf{k},\omega}(\mathbf{p})]$ with $G(p,\mu)=-\imath n_a f'_a(p)/[(\omega-kv\mu)p]$. Also the Maxwell operator only has diagonal elements with $\Lambda_{zz}=\Lambda_\parallel$ and $\Lambda_{xx}=\Lambda_{yy}=\Lambda_\perp$ with the longitudinal and transverse dispersion functions

$$\begin{pmatrix}\Lambda_\parallel(k,\omega)-1\\ \Lambda_\perp(k,\omega)-1+\frac{k^2c^2}{\omega^2}\end{pmatrix}=\frac{\pi}{\omega}\sum_a\omega^2_{p,a}\int_0^\infty dp\,\frac{p^3f'_a(p)}{\gamma}\times \int_{-1}^{1}\frac{d\mu}{\omega-kv\mu}\begin{pmatrix}2\mu^2\\ 1-\mu^2\end{pmatrix}, \quad (4)$$

with the plasma frequency $\omega_{p,a}=\sqrt{4\pi q_a^2n_a/m_a}$, $\gamma=(1-\beta^2)^{-1/2}$, and $\beta\equiv v/c$. For the stochastic force one notices with $\mathbf{E}(\mathbf{k},\omega)=(E_x,E_y,E_z)$ and $E_\perp=E_x\cos\phi+E_y\sin\phi$ that $K^*_\perp=q_a(\omega^*-kv\mu)E^*_\perp/\omega^*$ and $K^*_z=q_a[E^*_z+(kv/\omega^*)\sqrt{1-\mu^2}E^*_\perp]$, The Fourier-Laplace transformed electric field components are given by the solution of the wave equation (65) of paper I, in this case

$$E^*_{\parallel,\perp}=\frac{4\pi\imath}{\omega^*}\sum_b q_b\int d^3p'\,\delta N^{b0*}_{\mathbf{k},\omega}(\mathbf{p}')\frac{v'_{\parallel,\perp}}{\Lambda^*_{\parallel,\perp}}. \quad (5)$$

By applying Eq. (5) twice one obtains for the Fourier-Laplace transformed electric field correlation functions (Einstein summation convention adopted)

$$\begin{aligned}S_{ij}(\mathbf{k},\omega)&=\Re\left\langle E_iE^*_j\right\rangle(\mathbf{k},\omega)\\ &=\frac{16\pi^2\Lambda^{-1}_{im}\Lambda^{-1*}_{jn}}{|\omega|^2}\sum_{b,c}q_bq_c\int d^3p\int d^3p'v_mv'_n\,\mathcal{C}(\mathbf{p},\mathbf{p}')\\ &=\Re\,\frac{\imath\sum_b q_b^2n_b}{\pi^2|\omega|^2}\Lambda^{-1}_{im}\Lambda^{-1*}_{jn}\int d^3p\frac{v_mv_jf_b(p)}{\omega-kv\mu}\\ &=\Re\,\frac{\imath\sum_b\omega^2_{p,b}m_b\Lambda^{-1}_{im}\Lambda^{-1*}_{jn}}{\pi^2|\omega|^2}\times\\ &\left[\int_0^\infty dp\,p^2f_b(p)\int_{-1}^1\frac{d\mu}{\omega-kv\mu}\int_0^{2\pi}d\phi\,v_mv_j\right],\end{aligned} \quad (6)$$

where we used [19, 22]

$$\mathcal{C}(\mathbf{p},\mathbf{p}')\equiv\left\langle\delta N^{b0}_{\mathbf{k},\omega}(\mathbf{p})\delta N^{c0*}_{\mathbf{k},\omega}(\mathbf{p}')\right\rangle=\frac{\imath n_b\delta_{bc}f_b(p)\delta(\mathbf{v}'-\mathbf{v})}{(2\pi)^4(\omega-kv\mu)} \quad (7)$$

for the Fourier-Laplace transformed two-time correlation function of free-streaming uncorrelated particles in unmagnetized plasmas. With $\int_0^{2\pi}d\phi v_m(\phi)v_n(\phi)=\pi v^2[(1-\mu^2)(\delta_{m1}\delta_{n1}+\delta_{m2}\delta_{n2})+2\mu^2\delta_{m3}\delta_{n3}]$ the only non-vanishing electric field correlation functions are $S_\parallel=S_{zz}$ and $S_\perp=S_{xx}=S_{yy}$ given according to Eq. (6) by

$$S_{\parallel,\perp}=\Re\,\imath\sum_b\frac{\omega^2_{p,b}m_bc^2}{\pi(kc)^3|z\Lambda_{\parallel,\perp}|^2}\int_0^\infty dp\,p^2\beta f_b(p)\mathcal{I}_{\parallel,\perp}\left(\frac{z}{\beta}\right), \quad (8)$$

upon introducing the complex phase speed $z = \omega/kc$, i.e., $z/\beta = \omega/kv$, the integrals $\mathcal{I}_\parallel = \mathcal{I}_2(x)$ and $\mathcal{I}_\perp = \mathcal{I}_0(x) - \mathcal{I}_2(x)$ with $\mathcal{I}_n(x) = \int_{-1}^{1} d\mu\, \mu^n (x-\mu)^{-1}$. Both correlation functions (8) are completely determined by the isotropic particle distribution function $f_b(p)$. The index b instead of a indicates that in general the electric fluctuations entering the quasilinear momentum diffusion coefficient $\overline{D}_2$ for particles of sort a can be generated both by these particles themselves ($b = a$), referred to as self-confinement, and/or by other ($b \neq a$) charged particle populations in the considered plasma.

The drag and momentum diffusion terms are evaluated in detail in Supplementary Section A [23] as

$$\overline{D}_1(f_a) = \frac{\omega_{p,a}^2 m_a}{32\pi^4 n_a p^2}\frac{\partial}{\partial p}\Big[p^2 f_a(p)\int \frac{d^3k}{k}\int_Z dz\ \times \Re\left(\frac{\mathcal{I}_\parallel(z/\beta)}{(z\Lambda_\parallel)^*} + \frac{\mathcal{I}_\perp(z/\beta)}{(z\Lambda_\perp)^*}\right)\Big], \qquad (9)$$

and a corresponding, more lengthy, expression (A.12) for $\overline{D}_2$, where we made use of Eq. (8). It is important to emphasize that the derived drag term and momentum diffusion term (via the electric field correlation functions (8)) are based on the identical two-time correlation (7). The drag term (9) and the momentum diffusion term (A.12) are first important results of the present investigation.

Next, we focus on the self-confinement case $b = a$ and the high-frequency limit $|z| = |\omega|/kc \gg 1$ so that we ignore the influence of damped (negative $\Gamma < 0$) collective modes on the particle dynamics and approximate the dispersion functions as $\Lambda_\perp(k, z) \simeq 1$. It is well known [10, 12, 14–16] that in unmagnetized plasmas with isotropic particle distribution functions no growing collective modes with positive $\Gamma > 0$ exist. In the high-frequency limit and self-confinement case ($b = a$) the drag and momentum diffusion terms (9) and (A.12) then read

$$\overline{D}_1 = \frac{1}{p^2}\frac{d}{dp}[\beta H_1 p^2 f_a(p)], \quad \overline{D}_2 = \frac{1}{p^2}\frac{d}{dp}[H_2 p^2 f_a'(p)], \qquad (10)$$

with

$$H_1 = \frac{\omega_{p,a}^2 m_a}{32\pi^4 n_a \beta}\int \frac{d^3k}{k}\int_Z dz\,\Re\frac{\mathcal{I}_0(\frac{z}{\beta})}{z^*}, \qquad (11)$$

$$H_2 = \frac{\omega_{p,a}^4 m_a^2}{8\pi^2 n_a c\beta}\int \frac{d^3k}{k^3}\ \times \int_Z \frac{dz}{|z|^2}\Big\{[\Re\imath\mathcal{I}_0(\frac{z}{\beta})]\big[\int_0^\infty dp\, p^2\beta f_a(p)\Re\imath\mathcal{I}_0(\frac{z}{\beta})\big]\Big\}, \qquad (12)$$

where $\mathcal{I}_0(z/\beta) = 2\,\mathrm{arcoth}(z/\beta) = \mathrm{artanh}[2\beta R/(R^2 + I^2 + \beta^2)] - \imath\,\mathrm{artan}[2\beta I/(I^2 + R^2 - \beta^2)]$, provided $z = R + \imath I$. Within the same limit, where $\beta \leq 1 < I^2 + R^2 = |z|^2$ holds, the $\mathcal{I}_0(z/\beta)$ is continuous and analytic at all points in the complex z-plane and requires no analytic continuation. The frequency integrals $\int_Z dz$ appearing in H_1 and H_2 are calculated as 2β and $2\beta\int_0^\infty dp\, p^2\beta^2 f_a(p)$, respectively (Supplementary Sections B,C), leading to the momentum-independent

$$H_1 = \frac{\omega_{p,a}^2 m_a}{(2\pi)^4 n_a}\int \frac{d^3k}{k}, \quad H_2 = \frac{\omega_{p,a}^4 m_a^2 U_a}{(2\pi)^2 n_a c}\int \frac{d^3k}{k^3}, \qquad (13)$$

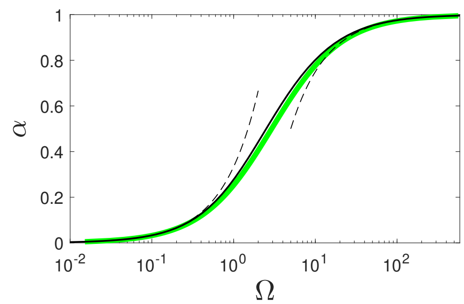

FIG. 1. **The dimensionless $\alpha(\Omega)$ (solid black) according to Eq.** (17). Black dashed lines are the exact asymptotics $\alpha = \Omega/3$ and $\alpha = 1 - 5(2\Omega)^{-1}$ for small and large Ω, respectively. The green line is $\Omega = 3\alpha(1-\alpha/6)/(1-\alpha)$, a superposition of the exact asymptotic expressions.

and $U_a = \int_0^\infty dp\, p^2\beta^2 f_a(p)$. The particle equilibrium condition (3) then reads in the high-frequency limit

$$\frac{1}{p^2}\frac{d}{dp}p^2\left[\frac{\Omega}{m_a c}\beta f_a(p) + f_a'(p)\right] = 0, \qquad (14)$$

with the positively valued, dimensionless ratio

$$\Omega = \frac{H_1}{H_2}m_a c = \frac{(k_{\max}^2 - k_{\min}^2)c^2}{(2\pi)^2\omega_{p,a}^2 U_a \ln(k_{\max}/k_{\min})}. \qquad (15)$$

With the requirement $\lim_{p\to\infty} f_a(p) = 0$ the Eq. (14) is solved by the relativistic Boltzmann-Jüttner distribution function (K_ν denotes a modified Bessel function)

$$f_a(p) = \frac{\Omega}{4\pi(m_a c)^3 K_2(\Omega)}\exp\left[-\Omega\gamma(p)\right], \qquad (16)$$

obeying the normalization requirement $\int d^3p\, f_a(p) = 1$. With this solution (16) one obtains $U_a = \Omega\Upsilon(\Omega)/4\pi K_2(\Omega)$ with $\Upsilon(\Omega) = \int_0^\infty dx[x^4\exp(-\Omega\sqrt{1+x^2})]/(1+x^2)$ which evaluates to $K_2(\Omega)/\Omega - K_1(\Omega) + \mathrm{Ki}_1(\Omega)$ where $\mathrm{Ki}_1(\Omega) = \int_\Omega^\infty dt K_0(t)$ [24]. Relation (15) then becomes

$$\alpha(\Omega) \equiv \frac{\Omega^2\Upsilon(\Omega)}{3K_2(\Omega)} = \frac{(k_{\max}^2 - k_{\min}^2)c^2}{3\pi\omega_{p,a}^2\ln\frac{k_{\max}}{k_{\min}}}. \qquad (17)$$

Most noteworthy $\alpha(\Omega)$ cannot get larger than unity (Fig. 1). Consequently, solutions of Eq. (17) are only possible if its right-hand side is also smaller than unity. The constant Ω defines the plasma temperature by $\Omega = m_a c^2/k_{\mathrm{B}}T_a$, because for large values of $\Omega \gg 1$ the distribution function (16) reduces to the nonrelativistic Maxwellian distribution $f_a(v) = (m_a/2\pi k_{\mathrm{B}}T_a)^{3/2}e^{-m_a v^2/2k_{\mathrm{B}}T_a}$.

The equilibrium distribution function (16) also results without the self-confinement assumption, however, with a different Ω depending on the origin of the electric field fluctuations $S_{\parallel,\perp}$. This is particularly true for heavier plasma particles, such as protons, which undergo momentum diffusion in the electric field fluctuations generated by the lighter electrons. The mechanism is self-regulating: besides generating the Boltzmann-Jüttner equilibrium distribution function it also

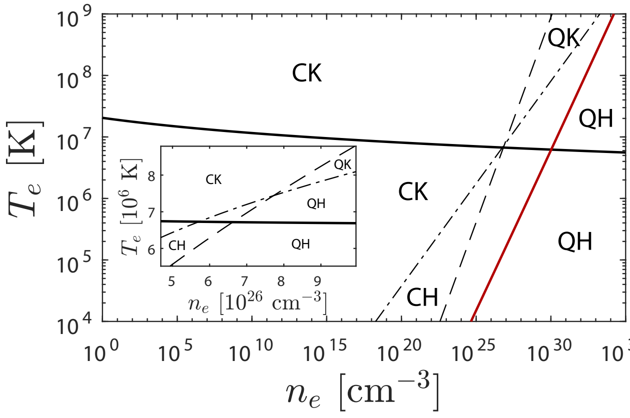

FIG. 2. **Resulting T_e as a function of the electron density n_e.** The regimes of classical collision-poor plasmas are separated from the regimes of quantum plasmas. The separating lines are given by $T_e = 8.84 \times 10^{-12} n_e^{2/3}$ (dashed) when the de Broglie wavelength equals the average interparticle distance, and $T_e = 8.11 \times 10^{-3} n_e^{1/3}$ (dot-dashed) to ensure plasma parameters $g < 1$. Abbreviations stand for classical (C), kinetic (K), hydodynamics (H), and quantum (Q), respectively. The inset indicates that the three curves for dilute plasmas do not intersect at a single point. For completeness we show as red curve the relation for dense stellar plasmas (Supplementary Section D, see also references [19, 25] therein) which lies outside the validity range of classical kinetic theory.

yields with Eq. (17) a defining equation for the plasma temperature provided the maximum wavenumber $k_{\max}$ is small enough. The maximum wavenumber $k_{\max}$ has to be regarded as the only free parameter of the presented analysis. What complicates the issue is that $k_{\max}$ itself can be temperature dependent.

Employing a standard choice of $k_{\max} = (\Omega r_e)^{-1}$ [19], involving the atomic classical electron radius r_e, together with the inverse Debye length $k_{\min} = \lambda_D^{-1}$, the resulting temperature-density relation $T_e(n_e)$ lies outside the parameter regime for the applicability of the classical kinetic theory (Supplementary Section D). For dilute plasmas we adopt $k_{\max} = \lambda_D^{-1} = \omega_{p,e}/v_{\mathrm{th,e}} = \omega_{p,e}\Omega_e^{1/2}/c$ being given by the inverse Debye length (for a motivation see Supplementary Section E, see also reference [26] therein). Here we identify the minimum wavenumber with $k_{\min} = 2\pi/L$, where L denotes the size of the considered astrophysical system. Equation (17) for electrons then reduces with $\alpha(\Omega) \simeq 1$ to

$$\frac{\Omega_e}{3\pi} \simeq \ln \frac{L}{2\pi\lambda_D} \simeq 47.6 + \ln\left(\frac{L}{\mathrm{kpc}}\right) - \ln T_e + \ln n_e, \quad (18)$$

where Ω_e depends only logarithmically weakly on n_e, T_e and size of the system scaled in kpc $= 3.086 \times 10^{21}$. With $\Omega_e = m_e c^2/k_{\mathrm{B}} T_e$ this Eq. (18) is solved in terms of the non-principal Lambert function as (Fig. 2)

$$T_e \simeq -\frac{6.29 \times 10^8}{W_{-1}(-1.337 \times 10^{-12}/n_e)}. \quad (19)$$

This result agrees remarkably well with the observed intracluster gas temperatures of $T_e \approx 10^{7-8}$ in the unmagnetized outer parts of clusters of galaxies [27, 28], where cooling effects by free-free photon emission can be neglected. For dilute plasmas almost independently of the density value the equilibrium temperature is $T_e \simeq 1.3 \times 10^7$ (Fig. 2). The estimate is valid for $n_e < 2 \times 10^{27}$ when the electron de Broglie wavelength is smaller than the average interparticle distance corresponding to thermal energies larger than the Fermi energy [1]. The characteristic relaxation time scale is given by $\tau = p/\beta = 4\pi^3 n_a c\gamma/(\omega_{p,e}k_{\max})^2 = 7 \times 10^{11}\gamma/n_e$. It is particularly short at high densities and remains momentum-independent for nonrelativistic electrons.

In conclusion, it has been demonstrated that an efficient thermalization mechanism in collision-poor unmagnetized plasmas results from the competition between the momentum losses of plasma particles by spontaneous emission of high-frequency non-collective fluctuations and the momentum diffusion of these particles in their self-generated fluctuating electric field. The mechanism is self-regulating but the resulting $T_e(n_e)$ relation is severely influenced by the adopted maximum wavenumber of the fluctuations. The mechanism provides $T_e \approx 10^7$ nearly independently of the plasma number density for super-miles long fluctuation wavelengths [29], and is applicable to plasmas with densities below 10^{27}. Future studies should investigate the influence of collective eigenmodes on this mechanism avoiding the high-frequency approximation.

[1] G. Manfredi, How to model quantum plasmas, in *Topics in Kinetic Theory*, Fields Institute Commun., Vol. 46, edited by T. Passot, C. Sulem, and P.-L. Sulem (American Mathematical Society, Providence, Rhode Island, United States, 2005) p. 263.
[2] Unless otherwise noted the units are in basic cgs and Kelvin.
[3] P. Helander and D. J. Sigmar, *Collisional Transport in Magnetized Plasmas* (Cambridge University Press, Cambridge, U.K, 2006).
[4] G. P. Zank, *Transport Processes in Space Physics and Astrophysics* (Springer, New York, USA, 2014).
[5] J. C. Kasper, A. J. Lazarus, and S. P. Gary, Geophys. Res. Lett. **29**, 20 (2002).
[6] S. D. Bale, J. C. Kasper, G. G. Howes, E. Quataert, C. Salem, and D. Sundkvist, Magnetic fluctuation power near proton temperature anisotropy instability thresholds in the solar wind, Phys. Rev. Lett. **103**, 211101 (2009).
[7] B. A. Maruca, J. C. Kasper, and S. D. Bale, What are the relative roles of heating and cooling in generating solar wind temperature anisotropies?, Phys. Rev. Lett. **107**, 201101 (2011).
[8] T. A. Bowen, B. D. Chandran, J. Squire, S. D. Bale, D. Duan, K. G. Klein, D. Larson, A. Mallet, M. D. McManus, R. Meyrand, *et al.*, In situ signature of cyclotron resonant heating in the solar wind, Phys. Rev. Lett. **129**, 165101 (2022).
[9] R. Schlickeiser, M. M. Martinovic, and P. H. Yoon, Phys. Plas-

mas **28**, 052110 (2021).
[10] T. Felten and R. Schlickeiser, Spontaneous electromagnetic fluctuations in unmagnetized plasmas. vi. transverse, collective mode for arbitrary distribution functions, Phys. Plasmas **20**, 104502 (2013).
[11] R. Schlickeiser and P. H. Yoon, Quasilinear theory of general electromagnetic fluctuations in unmagnetized plasmas, Phys. Plasmas **21**, 092102 (2014).
[12] I. B. Bernstein, Waves in a plsma in a magnetic field, Phys. Rev. **109**, 10 (1958).
[13] C. S. Gardner, Bound on the energy available from a plasma, Phys. Fluids **6**, 839 (1963).
[14] N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill, New York, United States, 1973).
[15] S. P. Gary, *Theory of Space Plasma Microinstabilities* (Cambridge University Press, Cambridge, U.K., 1993).
[16] R. Schlickeiser and M. Kneller, Relativistic kinetic theory of waves in isotropic plasmas, J. Plasma Phys. **57**, 709 (1997).
[17] R. Schlickeiser and P. H. Yoon, Quasilinear theory of general electromagnetic fluctuations including discrete particle effects for magnetized plasmas: General analysis, Phys. Plasmas **29**, 092105 (paper I) (2022).
[18] Y. L. Klimontovich, *Kinetic theory of nonideal gases and nonideal plasma* (Pergamon, Oxford, U.K., 1982).
[19] P. H. Yoon, *Classical Kinetic Theory of Weakly Turbulent Nonlinear Plasma Processes* (Cambridge University Press, Cambridge, U.K., 2019).
[20] M. Born and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation. Interference and Diffraction of Light* (Pergamon Press, Oxford, United States, 1965).
[21] D. H. Froula, S. H. Glenzer, N. C. Luhmann, and J. Sheffield, *Plasma Scattering of Electromagnetic Radiation* (Elsevier, Amsterdam, The Netherlands, 2011).
[22] R. Schlickeiser and P. H. Yoon, Spontaneous electromagnetic fluctuations in unmagnetized plasmas I: General theory and nonrelativistic limit, Phys. Plasmas **19**, 022105 (2012).
[23] See supplemental material at [url will be inserted by publisher] for drag and momentum diffusion terms, the complex ω-integration, evaluation of frequency integrals, dense stellar plasmas, and comment on $k_{\max}$.
[24] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover Publications, New York, 1972).
[25] J. N. Bahcall, *Neutrino Astrophysics* (Cambridge University Press, Cambridge, U.K., 1989).
[26] R. Schlickeiser, U. Kolberg, and P. H. Yoon, Primordial plasma fluctuations. i. magnetization of the early universe by dark aperiodic fluctuations in the past myon and prior electron–positron annihilation epoch, Astrophys. J. **857**, 29 (2018).
[27] A. C. Fabian, P. E. J. Nulsen, and C. R. Canizares, Cooling flows in clusters of galaxies, Nature **310**, 733 (1984).
[28] C. L. Sarazin, *X-ray Emission from Clusters of Galaxies* (Cambridge University Press, Cambridge, U.K., 1988).
[29] Such very long wavelengths make it difficult to identify them in particle-in-cell plasma simulations of limited size.

SUPPLEMENTARY INFORMATION

Quasilinear thermalization of collision-poor plasmas by noncollective fluctuations

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A: Drag and momentum diffusion term

If $\mathbf{R} = (R_x, R_y, R_z)$ represents either $\langle \delta N_a^0 \delta \mathbf{K}^* \rangle$ and $\langle A \delta \mathbf{K}^* \rangle$ in spherical momentum coordinates one obtains for the pitch-angle and gyrophase average

$$\frac{1}{4\pi}\int_{-1}^{1} d\mu \int_0^{2\pi} d\phi \frac{\partial}{\partial \mathbf{p}} \times \mathbf{R} = \frac{1}{4\pi}\int_{-1}^{1} d\mu \int_0^{2\pi} d\phi \left(\frac{\partial}{p^2 \partial p} p^2 [\sqrt{1-\mu^2} R_\perp + \mu R_z]\right), \tag{A.1}$$

with $R_\perp = R_x \cos\phi + R_y \sin\phi$. In order to determine the drag term one infers with the stochastic forces and the average (7)

$$\begin{aligned}\left\langle E_\perp^* \delta N_{\mathbf{k},\omega}^{a0} \right\rangle &= \left\langle E_x^* \delta N_{\mathbf{k},\omega}^{a0} \right\rangle \cos\phi + \left\langle E_y^* \delta N_{\mathbf{k},\omega}^{a0} \right\rangle \sin\phi \\ &= \frac{4\pi\imath}{(\omega\Lambda_\perp)^*} \sum_b q_b \int d^3p' (v'_x \cos\phi + v'_y \sin\phi) \left\langle \delta N_{\mathbf{k},\omega}^{a0}(\mathbf{p}) \delta N_{\mathbf{k},\omega}^{b0*}(\mathbf{p}') \right\rangle \\ &= -\frac{q_a n_a v f_a}{4\pi^3 (\omega - kv\mu)\omega^*} \frac{\sqrt{1-\mu^2}}{\Lambda_\perp^*},\end{aligned} \tag{A.2}$$

$$\left\langle E_z^* \delta N_{\mathbf{k},\omega}^{a0*} \right\rangle = -\frac{q_a n_a v f_a(p)}{4\pi^3 \omega^* (\omega - kv\mu)} \frac{\mu}{\Lambda_\parallel^*}, \tag{A.3}$$

implying

$$\begin{pmatrix} \left\langle K_\perp^* \delta N_{\mathbf{k},\omega}^{a0} \right\rangle \\ \left\langle K_z^* \delta N_{\mathbf{k},\omega}^{a0} \right\rangle \end{pmatrix} = -\frac{\omega_{p,a}^2 m_a v f_a(p)}{16\pi^4 \omega^* (\omega - kv\mu)} \begin{pmatrix} (1 - \frac{kv\mu}{\omega^*}) \frac{\sqrt{1-\mu^2}}{\Lambda_\perp^*} \\ \frac{\mu}{\Lambda_\parallel^*} + \frac{kv(1-\mu^2)}{\omega^* \Lambda_\perp^*} \end{pmatrix}. \tag{A.4}$$

With Eq. (A.1) the drag term (3) then becomes

$$\begin{aligned}\widetilde{D}_1(f_a) = \frac{1}{2}\int_{-1}^{1} d\mu\, \mathcal{D}_1(f_a) &= \frac{\omega_{p,a}^2 m_a}{32\pi^4 n_a p^2} \frac{\partial}{\partial p} \left[p^2 v f_a(p) \int d^3k \int d\omega\, \Re \frac{1}{\omega^*} \int_{-1}^{1} \frac{d\mu}{\omega - kv\mu} \left(\frac{\mu^2}{\Lambda_\parallel^*} + \frac{1-\mu^2}{\Lambda_\perp^*} \right) \right] \\ &= \frac{\omega_{p,a}^2 m_a}{32\pi^4 n_a p^2} \frac{\partial}{\partial p} \left[p^2 f_a(p) \int \frac{d^3k}{k} \int dz\, \Re \frac{1}{z^*} \int_{-1}^{1} \frac{d\mu}{\frac{z}{\beta} - \mu} \left(\frac{\mu^2}{\Lambda_\parallel^*} + \frac{1-\mu^2}{\Lambda_\perp^*} \right) \right].\end{aligned} \tag{A.5}$$

As explained in Supplementary Section B the time average of Eq. (A.5) determines the ω-integral of any complex quantity $g(\omega) = g(\omega_R + \imath\Gamma)$ as

$$\int d\omega\, g(\omega) = \int_Z d\omega\, g(\omega) = \lim_{\Gamma \to 0} \frac{\Gamma}{\pi} \int_{-\infty}^{\infty} d\omega_R\, g(\omega_R + \imath\Gamma), \tag{A.6}$$

or in terms of the phase speed $z = \omega/kc = R + \imath I$

$$\int dz\, g(z) = \int_Z dz\, g(z) = \lim_{I \to 0} \frac{I}{\pi} \int_{-\infty}^{\infty} dR\, g(R + \imath I), \tag{A.7}$$

which we denote by the index Z in the respective integration limits. Consequently, one obtains for the time-averaged drag term (3)

$$\overline{D}_1(f_a) = \frac{\omega_{p,a}^2 m_a}{32\pi^4 n_a p^2}\frac{\partial}{\partial p}\left[p^2 f_a(p)\int \frac{d^3k}{k}\int_Z dz\,\Re\int_{-1}^{1}\frac{d\mu}{\frac{z}{\beta}-\mu}\left(\frac{\mu^2}{(z\Lambda_\parallel(k,z))^*}+\frac{1-\mu^2}{(z\Lambda_\perp(k,z))^*}\right)\right]. \tag{A.8}$$

The determination of the momentum diffusion term is more involved. First, with $A_{\mathbf{k},\omega}(\mathbf{p}) = G(p,\mu)[\mathbf{p}\cdot\mathbf{K}_{\mathbf{k},\omega}(\mathbf{p})]$, where $G(p,\mu) = -\imath n_a(\partial f_a/\partial p)/(\omega - kv\mu)p$, one obtains in spherical momentum coordinates

$$\begin{aligned}
&\int_0^{2\pi} d\phi\,\frac{\partial}{\partial \mathbf{p}}\cdot\langle \mathbf{K}^*(\mathbf{p})A_{\mathbf{k},\omega}(\mathbf{p})\rangle = \int_0^{2\pi} d\phi\,\frac{\partial}{\partial \mathbf{p}}\cdot\left[\langle(\mathbf{p}\cdot\mathbf{K})\mathbf{K}^*\rangle\, G(p,\mu)\right]\\
&= \int_0^{2\pi} d\phi\,\frac{1}{p^2}\frac{\partial}{\partial p}p^3G(p,\mu)\left[\langle K_\perp K_\perp^*\rangle(1-\mu^2)+\langle K_zK_z^*\rangle\mu^2+\mu\sqrt{1-\mu^2}(\langle K_\perp K_z^*\rangle+\langle K_zK_\perp^*\rangle)\right]\\
&\quad+\int_0^{2\pi} d\phi\,\frac{\partial}{\partial \mu}G(p,\mu)\left[\mu(1-\mu^2)[\langle K_zK_z^*\rangle-\langle K_\perp K_\perp^*\rangle]+(1-\mu^2)^{3/2}\langle K_\perp K_z^*\rangle-\mu^2\sqrt{1-\mu^2}\langle K_zK_\perp^*\rangle\right]\\
&= \int_0^{2\pi} d\phi\Big[\frac{1}{p^2}\frac{\partial}{\partial p}p^3G(p,\mu)[\langle K_\perp K_\perp^*\rangle(1-\mu^2)+\langle K_zK_z^*\rangle\mu^2]+\frac{\partial}{\partial\mu}G(p,\mu)\mu(1-\mu^2)[\langle K_zK_z^*\rangle-\langle K_\perp K_\perp^*\rangle]\Big],
\end{aligned} \tag{A.9}$$

because $\int_0^{2\pi} d\phi\langle K_\perp K_z^*\rangle = \int_0^{2\pi} d\phi\langle K_zK_\perp^*\rangle = 0$. With

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} d\phi\langle K_\perp^2\rangle &= \frac{q_a^2|\omega-kv\mu|^2}{|\omega|^2}\frac{S_{xx}+S_{yy}}{2},\\
= \frac{1}{2\pi}\int_0^{2\pi} d\phi\langle K_zK_z^*\rangle &= q_a^2\left[S_{zz}+\frac{k^2v^2(1-\mu^2)}{|\omega|^2}\frac{S_{xx}+S_{yy}}{2}\right],
\end{aligned} \tag{A.10}$$

inserted in Eq. (A.9) and averaging the result over μ provides for the momentum diffusion coefficient in Eq. (3)

$$\begin{aligned}
\widetilde{D}_2(f_a) = \frac{1}{2}\int_{-1}^{1} d\mu\,\mathcal{D}_2(f_a) &= \frac{\omega_{p,a}^2 m_a}{8\pi n_a}\int d^3k\int d\omega\int_{-1}^{1} d\mu\frac{1}{p^2}\frac{\partial}{\partial p}p^2\Re\,\frac{\imath}{\omega-kv\mu}\times\\
&\quad\Big[S_{zz}(k,\omega)\mu^2+S_\perp(k,\omega)(1-\mu^2)\frac{|\omega-kv\mu|^2+k^2v^2\mu^2}{|\omega|^2}\Big]f_a'(p)\\
&= \frac{\omega_{p,a}^2 m_a}{8\pi n_a}\int d^3k\int dz\int_{-1}^{1} d\mu\frac{1}{p^2}\frac{\partial}{\partial p}\frac{p^2}{\beta}\Re\,\frac{\imath}{\frac{z}{\beta}-\mu}\times\\
&\quad\Big[S_{zz}(k,z)\mu^2+S_\perp(k,z)(1-\mu^2)[|1-\frac{\beta\mu}{z}|^2+\frac{\beta^2\mu^2}{|z|^2}]\Big]f_a'(p)
\end{aligned} \tag{A.11}$$

where $f_a'(p) = df_a/dp$. Inserting the electric field correlation functions (8) and applying the prescription (A.7) then leads to

$$\begin{aligned}
\overline{D}_2(f_a) &= \frac{\omega_{p,a}^2 m_a}{8\pi n_a}\int d^3k\int_Z dz\int_{-1}^{1} d\mu\frac{1}{p^2}\frac{\partial}{\partial p}\frac{p^2}{\beta}\Re\,\frac{\imath}{\frac{z}{\beta}-\mu}\Big[S_\parallel(k,z)\mu^2+S_\perp(k,z)(1-\mu^2)[|1-\frac{\beta\mu}{z}|^2+\frac{\beta^2\mu^2}{|z|^2}]\Big]f_a'(p)\\
&= \frac{\omega_{p,a}^2 m_a c^2}{8\pi^2 n_a}\sum_b \omega_{p,b}^2 m_b\int\frac{d^3k}{(kc)^3}\int_Z\frac{dz}{|z|^2}\int_{-1}^{1}d\mu\frac{1}{p^2}\frac{\partial}{\partial p}\frac{p^2}{\beta}\Re\,\frac{\imath}{\frac{z}{\beta}-\mu}\Re\imath\times\\
&\quad\left\{\frac{\mu^2}{|\Lambda_\parallel|^2}\int_0^\infty dp\,p^2\beta f_b(p)\mathcal{I}_\parallel\left(\frac{z}{\beta}\right)+\frac{1-\mu^2}{|\Lambda_\perp|^2}[|1-\frac{\beta\mu}{z}|^2+\frac{\beta^2\mu^2}{|z|^2}]\int_0^\infty dp\,p^2\beta f_b(p)\,\mathcal{I}_\perp\left(\frac{z}{\beta}\right)\right\}f_a'(p).
\end{aligned} \tag{A.12}$$

B: Complex ω-integration

Let $b(\mathbf{r},t)$ represent one of the stochastic force components $\delta\mathbf{K}(\mathbf{r},\mathbf{p},t)$ and let $a(\mathbf{r},t)$ represent either $\delta N^0(\mathbf{r},\mathbf{p},t)$ or $A(\mathbf{r},\mathbf{p},t)$. We introduce the Fourier-Laplace (FL) transformations of all fluctuating quantities with respect to positional coordinates and time with complex $\omega = \omega_R + \imath\Gamma$ as

$$(2\pi)^4 \begin{pmatrix} a_{\mathbf{k},\omega} \\ b_{\mathbf{k},\omega} \end{pmatrix} = \Re \int_{-\infty}^{\infty} d^3r \int_0^{\infty} dt \; e^{-\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)} \begin{pmatrix} a(\mathbf{r},t) \\ b(\mathbf{r},t) \end{pmatrix}. \tag{B.1}$$

The respective inverse FL transform then is

$$\begin{pmatrix} a(\mathbf{r},t) \\ b(\mathbf{r},t) \end{pmatrix} = \Re \int d^3k \int_L d\omega \; e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)} \begin{pmatrix} a_{\mathbf{k},\omega} \\ b_{\mathbf{k},\omega} \end{pmatrix}. \tag{B.2}$$

In order for the Laplace inversion (B.1) to exist the imaginary part of the frequency $\Gamma = \Im(\omega) > 0$ has to be positive so that the FL transformed equations originally only hold in the positive complex frequency plane. Proper analytical continuation into the negative complex frequency plane is required to derive results valid in the whole complex frequency plane. The index L, standing for the Landau contour, in the frequency integral of Eq. (B.2) has to allow for the proper analytical continuation.

In quasi-stationary plasmas one is interested in times averages for times T large compared to microscopic fluctuations times of a quantity

$$|f(t)|^2 = \lim_{T\to\infty} \int_{-T/2}^{T/2} dt\, f(t) f^*(t) = \lim_{T\to\infty} \int_{-\infty}^{\infty} dw \frac{|f(w)|^2}{2\pi T} = \int_{-\infty}^{\infty} dw\, S(w) \tag{B.3}$$

according to Parseval's theorem, where w refers to the frequency of a *Fourier transform* also for the time variable t, and where the spectral function is defined as (Froula *et al.* [21], Supplementary Section A)

$$S(w) = \lim_{T\to\infty} \frac{|f(w)|^2}{2\pi T}. \tag{B.4}$$

It is the Fourier transform of the autocorrelation function

$$C(\tau) = \lim_{T\to\infty} \int_{-T/2}^{T/2} dt\, f(t) f^*(t+\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw\, e^{\imath w\tau} S(w), \quad S(w) = \int_{-\infty}^{\infty} d\tau\, e^{-\imath w\tau} C(\tau). \tag{B.5}$$

Next we investigate the relation of the spectral function (B.4) defined in terms of Fourier time transforms to the corresponding quantity calculated with the Laplace time transform with $\omega = \omega_R + \imath\Gamma$. With Eq. (B.1)

$$\langle f(\omega) f^*(\omega) \rangle = \int_0^{\infty} dt\, e^{(\imath\omega_R - \Gamma)t} \int_0^{\infty} dt'\, e^{-(\imath\omega_R+\Gamma)t'} \langle f(t) f^*(t') \rangle. \tag{B.6}$$

With $t' = t + \tau$

$$\begin{aligned} \langle f(\omega) f^*(\omega) \rangle &= \int_0^{\infty} dt\, e^{-2\Gamma t} \int_{-t}^{\infty} d\tau\, e^{-(\imath\omega_R+\Gamma)\tau} \langle f(t) f^*(t) \rangle \\ &= \left[\int_0^{\infty} dt\, e^{-2\Gamma t} \int_0^{\infty} d\tau\, e^{-(\imath\omega_R+\Gamma)\tau} + \int_0^{\infty} dt\, e^{-2\Gamma t} \int_{-t}^{0} d\tau\, e^{-(\imath\omega_R+\Gamma)\tau} \right] \langle f(t) f^*(t+\tau) \rangle. \end{aligned} \tag{B.7}$$

For stationary plasmas $\langle f(t) f^*(t+\tau) \rangle$ is not a function of t, so that partial integration of the second integral with respect to t yields

$$\begin{aligned} \int_0^{\infty} dt\, e^{-2\Gamma t} \int_{-t}^{0} d\tau\, e^{-(\imath\omega_R+\Gamma)\tau}) &= \frac{1}{2\Gamma} \int_0^{\infty} dt e^{(\imath\omega_R-\Gamma)t} - \Big[\frac{e^{-2\Gamma t}}{2\Gamma} \int_{-t}^{0} d\tau\, e^{-(\imath\omega_R+\Gamma)\tau} \Big]_0^{\infty} \\ &= \frac{1}{2\Gamma} \int_0^{\infty} dt e^{(\imath\omega_R-\Gamma)t}. \end{aligned} \tag{B.8}$$

Together with $\int_0^{\infty} dt e^{-2\Gamma t} = 1/2\Gamma$ in the first integral one obtains for Eq. (B.7)

$$\langle f(\omega) f^*(\omega) \rangle = \frac{1}{2\Gamma} [\int_0^{\infty} d\tau\, e^{-(\imath\omega_R+\Gamma)\tau} + \int_0^{\infty} d\tau e^{(\imath\omega_R-\Gamma)\tau}] \langle f(t) f^*(t+\tau) \rangle. \tag{B.9}$$

Multiplying by 2Γ and taking the limit $\Gamma \to 0$ leads to

$$\lim_{\Gamma\to 0} 2\Gamma \langle f(\omega) f^*(\omega) \rangle = \int_0^{\infty} d\tau\, [e^{-\imath\omega_R\tau} + e^{\imath\omega_R\tau}] \langle f(t) f^*(t+\tau) \rangle = \int_{-\infty}^{\infty} d\tau e^{\imath\omega_R\tau} \langle f(t) f^*(t+\tau) \rangle, \tag{B.10}$$

derived before [21]. The right-hand side of Eq. (B.10) corresponds exactly to the Fourier transform with $w = \omega_R$. Fourier inversion of Eq. (B.10) then provides

$$\langle f(t) f^*(t+\tau)\rangle = \lim_{\Gamma\to 0} 2\Gamma \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega_R\, e^{-\imath\omega_R\tau}\langle |f(\omega)|^2\rangle. \tag{B.11}$$

For the special case of $\tau = 0$ this implies

$$\langle |f(t)|^2\rangle = \lim_{\Gamma\to 0} 2\Gamma \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega_R\, \langle |f(\omega)|^2\rangle. \tag{B.12}$$

Comparing with Eq. (B.3) then shows that the spectral function (B.4) is given in terms of the Laplace time transform by

$$S(\omega) = S(\omega_R + \imath\Gamma) = \lim_{\Gamma\to 0}\frac{\Gamma}{\pi}\langle |f(\omega)|^2\rangle. \tag{B.13}$$

The average $\langle |f(\omega)|^2\rangle$ occurs when considering ensemble average calculated with the FL-transforms (B.1)

$$\langle a(\mathbf{r},t) b^*(\mathbf{r},t)\rangle = \int d^3k \int d^3k' \left\langle a_{\mathbf{k},\omega} b^*_{\mathbf{k}',\omega}\right\rangle e^{\imath(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}-\imath(\omega+\omega^*)t} = \int d^3k \left\langle a_{\mathbf{k},\omega} b^*_{\mathbf{k},\omega}\right\rangle e^{-\imath(\omega+\omega^*)t}, \tag{B.14}$$

where we used that fluctuations at different values of the wavenumber vector are uncorrelated $\langle a_{\mathbf{k},\omega} b^*_{\mathbf{k}',\omega'}\rangle = \delta(\mathbf{k}-\mathbf{k}')\langle a_{\mathbf{k},\omega} b^*_{\mathbf{k}',\omega'}\rangle$, Consequently, we identify

$$\langle |f(\omega)|^2\rangle = \int d^3k \left\langle a_{\mathbf{k},\omega} b^*_{\mathbf{k},\omega}\right\rangle, \tag{B.15}$$

so that the spectral function (B.11) is related to the FL-transformed ensemble average as

$$\int d^3k \int d\omega S(\mathbf{k},\omega) = \int d^3k \lim_{\Gamma\to 0}\frac{\Gamma}{\pi}\int d\omega_R \left\langle a_{\mathbf{k},\omega} b^*_{\mathbf{k},\omega}\right\rangle. \tag{B.16}$$

Applied to Eqs. (11) we have to evaluate the ω-integration as

$$\int d\omega V(\omega) = \lim_{\Gamma\to 0}\frac{\Gamma}{\pi}\int_{-\infty}^{\infty} d\omega_R\, V(\omega_R + \imath\Gamma), \tag{B.17}$$

or in terms of the phase speed $z = \omega/kc = R + \imath I$

$$\int dz V(z) = \lim_{I\to 0}\frac{I}{\pi}\int_{-\infty}^{\infty} dR\, V(R + \imath I). \tag{B.18}$$

C: Evaluation of frequency integrals

Here we evaluate the double integrals

$$u_1 = \int_Z dz\, \Re\frac{1}{z^*}\mathcal{I}_0(z/\beta) = \int_Z dz\, \Re\frac{1}{z^*}\int_{-1}^{1}\frac{d\mu}{\frac{z}{\beta}-\mu}, \tag{C.1}$$

and

$$u_2 = \int_Z dz\, \frac{1}{|z|^2}\left[\Re\imath\mathcal{I}_0(z/\beta_x)\right]\Big[\int_0^{\infty} dy\, y^2\beta_y f_a(y)\Re\imath\mathcal{I}_0(z/\beta_y)\Big], \tag{C.2}$$

where $\beta_x = \beta(x)$ and $\beta_y = \beta(y)$. With

$$\Re\frac{\mathcal{I}_0(z/\beta)}{z^*} = \frac{1}{R^2+I^2}\Big[R\,\text{artanh}\,\frac{2\beta R}{R^2+I^2+\beta^2} + I\text{arctan}\frac{2\beta I}{R^2+I^2-\beta^2}\Big] \tag{C.3}$$

we obtain for the first integral (11) with the prescription (A.7)

$$u_1 = \int_Z dz V_1(z) = \lim_{I\to 0} \frac{I}{\pi} J_1(I),$$
$$J_1(I) = \int_{-\infty}^{\infty} \frac{dR}{R^2+I^2}\big[R\,\mathrm{artanh}\,\frac{2\beta R}{R^2+I^2+\beta^2} + I\,\mathrm{arctan}\frac{2\beta I}{R^2+I^2-\beta^2}\big]. \tag{C.4}$$

For small velocities $\beta \ll 1$ we approximate

$$\mathrm{artanh}\,\frac{2\beta I x}{I^2(1+x^2)+\beta^2} \simeq \frac{2\beta x}{I(1+x^2)}, \quad \mathrm{arctan}\frac{2\beta I}{I^2(1+x^2)+\beta^2} \simeq \frac{2\beta}{I(1+x^2)}, \tag{C.5}$$

so that

$$J_1(I) \simeq 2\beta \int_{-\infty}^{\infty} \frac{dR}{R^2+I^2} = \frac{2\pi\beta}{I}, \tag{C.6}$$

where we used Eq. (A.1.15) of [21]:

$$\int_{-\infty}^{\infty} dx \frac{g(x)}{(x-a)^2+b^2} = \frac{\pi}{b} g(a). \tag{C.7}$$

Hence one finds for the limit (C.4)

$$u_1 = \lim_{I\to 0}\left(\frac{I}{\pi}J_1\right) = 2\beta. \tag{C.8}$$

Likewise, with $\beta(x) = \beta_x$ and $\beta(y) = \beta_y$ we infer the integrand of Eq. (C.2)

$$V_2(z,x) = \frac{1}{|z|^2}[\Re\imath\mathcal{I}_0(z/\beta)]\big[\int_0^{\infty} dx\, x^2\beta f_a(x)\Re\imath\mathcal{I}_0(z/\beta)\big] = \frac{1}{|z|^2}[\Re\imath\mathcal{I}_0(z/\beta_x)]\big[\int_0^{\infty} dy\, y^2\beta_y f_a(y)\Re\imath\mathcal{I}_0(z/\beta_y)\big]$$
$$= \int_0^{\infty} dy\, y^2\beta_y f_a(y)\frac{1}{|z^2|}[\Re\imath\mathcal{I}_0(z/\beta_x)]\Re\imath\mathcal{I}_0(z/\beta_y) = \int_0^{\infty} dy\, y^2\beta_y f_a(y) v_2(z,x,y), \tag{C.9}$$

where

$$v_2(z,x,y) = \frac{1}{|z^2|}[\Re\imath\mathcal{I}_0(z/\beta_x)]\Re\imath\mathcal{I}_0(z/\beta_y). \tag{C.10}$$

According to the prescription (A.7) we first evaluate

$$\int_{-\infty}^{\infty} dR\, v_2(R+\imath I,x,y) = \int_{-\infty}^{\infty}\frac{dR}{R^2+I^2}\,\mathrm{arctan}\left(\frac{2\beta_x I}{R^2+I^2-\beta_x^2}\right)\mathrm{arctan}\left(\frac{2\beta_y I}{R^2+I^2-\beta_y^2}\right). \tag{C.11}$$

With the approximation (C.5) and Eq. (C.6) this leads to

$$\int_{-\infty}^{\infty} dR\, v_2(R+\imath I,x,y) \simeq 4\beta_x\beta_y I^2\int_{-\infty}^{\infty}\frac{dR}{(R^2+I^2)^2} = -2\beta_x\beta_y I\frac{\partial}{\partial I}\int_{-\infty}^{\infty}\frac{dR}{R^2+I^2}$$
$$= -2\beta_x\beta_y I\frac{d}{dI}\frac{\pi}{I} = \frac{2\pi\beta_x\beta_y}{I}, \tag{C.12}$$

so that

$$\lim_{I\to 0}\frac{I}{\pi}\int_{-\infty}^{\infty} dR\, v_2(R+\imath I,x,y) = 2\beta_x\beta_y. \tag{C.13}$$

Consequently, the integral (C.2) becomes

$$u_2 = \int dz\, V_2(z,x) = 2\beta_x\int_0^{\infty} dy\, y^2\beta_y^2 f_a(y). \tag{C.14}$$

D: Dense stellar plasmas

With the standard choice of $k_{\rm max} = (\Omega r_e)^{-1}$ (Yoon [19], p. 114ff.), involving the atomic classical electron radius $r_e = e^2/m_e c^2 = 2.82 \times 10^{-13}$, together with $k_{\rm min} = \lambda_D^{-1}$ with the Debye length $\lambda_D = 6.9(T_e/n_e)^{1/2}$ provides for Eq. (17) at nonrelativistic temperatures $\alpha \simeq 1 = 6.1 \times 10^{17}/[\Omega\sqrt{n_e \ln(3/g)}]$, so that

$$T_e = 9.7 \times 10^{-9} n_e^{1/2} \sqrt{8.3 - 0.5 \ln n_e + 1.5 \ln T_e} \simeq 2.8 \times 10^{-8} n_e^{1/2}. \tag{D.1}$$

This temperature is orders of magnitude too small for dilute space plasmas, but may be interesting for the interior of solar-type stars where according to the standard solar model [25] the central density is $n_e \simeq 10^{26}$ so that $T_e \simeq 2.8 \times 10^5$. This stellar estimate also shown in Fig. 2. However, this result is preliminary as the resulting stellar temperature-density relation, calculated with classical kinetic theory, no longer lies in the classical kinetic regime as also clearly visible in Fig. 2.

E: Comment on $k_{\rm max}$

Equation (17) may also be regarded as a determining Eq. for $k_{\rm max}$ if the plasma Maxwellization is provided by a different mechanism. This is certainly the case for the early universe after the myon annihilation epoch but before the electron–positron annihilation epoch, i.e., at temperatures 1 MeV $< k_{\rm B} T_e <$ 210 MeV, corresponding to $\Omega_e \in [0.002, 0.5]$. At this epoch at cosmological redshifts $z_c \simeq 10^{10}$, the frequent annihilation and pair production processes of electron-positron pairs in thermal equilibrium with photons with the blackbody Planckian distribution function were responsible for the Maxwellization of the pairs. Using pair densities [26] of $n_e \simeq 10^{33}\Omega_e^3$, $\Omega_e = 0.5$, implying $\alpha = 0.17$ then yields from Eq. (17) with the relativistic pair plasma frequency $\omega_p^2 = 2\omega_{p,e}^2 \Omega_e$

$$k_{\rm max}(z_c) = \frac{\omega_{p,e}}{c} \sqrt{6\pi\Omega_e \alpha(\Omega_e) \ln(k_{\rm max}/k_{\rm min})} \simeq 7.5 \cdot 10^{10} \tag{E.1}$$

for the maximum wavenumber of noncollective fluctuations at this epoch, where we adopted $\ln(k_{\rm max}/k_{\rm min}) = 8$ for this estimate. The subsequent Hubble expansion then would generate for the present-day maximum wavenumber in dilute plasmas

$$k_{\rm max} = \frac{k_{\rm max}(z_c)}{1 + z_c} = 7.5 \tag{E.2}$$

which is of the same order of magnitude with the choosen value λ_D^{-1} agreeing with it for $T_e = 3.6 \cdot 10^{-4} n_e$. Hence the early universe history of all dilute space plasmas may motivate the chosen small present-day maximum wavenumber.